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1997 J. Phys. A: Math. Gen. 30 7269

(http://iopscience.iop.org/0305-4470/30/20/025)

J. Phys. A: Math. Gen. **30** (1997) 7269–7275. Printed in the UK PII: S0305-4470(97)80866-9

New generalizations of the integrable problems in rigid body dynamics

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Received 30 December 1996, in final form 15 May 1997

Abstract. We consider the general problem of motion of a rigid body about a fixed point under the action of an axisymmetric combination of potential and gyroscopic forces. We introduce six cases of this problem which are completely integrable for arbitrary initial conditions. The new cases generalize by several parameters all, but one, of the known results in the subject of rigid body dynamics. Namely, we generalize all the results due to Euler, Lagrange, Clebsch, Kovalevskaya, Brun and Lyapunov and also their subsequent generalizations by Rubanovsky and the present author.

1. Introduction

It is well known that integrable systems are a rare exception in dynamics. It is thus of great importance to construct as many integrable problems as possible and to consider every one of them in the most general possible form. The problem considered here is the general problem of motion of a rigid body about a fixed point under the action of a combination of conservative axisymmetric potential and gyroscopic forces. The equations of motion for this problem can be written in the Euler–Poisson form [1]

$$\begin{aligned}
&A\dot{p} + (C-B)qr + q\mu_3 - r\mu_2 = \gamma_2\frac{\partial V}{\partial\gamma_3} - \gamma_3\frac{\partial V}{\partial\gamma_2}\\
&B\dot{q} + (A-C)pr + r\mu_1 - p\mu_3 = \gamma_3\frac{\partial V}{\partial\gamma_1} - \gamma_1\frac{\partial V}{\partial\gamma_3}\\
&C\dot{r} + (B-A)pq + p\mu_2 - q\mu_1 = \gamma_1\frac{\partial V}{\partial\gamma_2} - \gamma_2\frac{\partial V}{\partial\gamma_1}\\
&\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0 \qquad \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0 \qquad \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0
\end{aligned} \tag{1}$$

where A, B, C are the principal moments of inertia, p, q, r are the components of the angular velocity of the body and γ_1, γ_2, γ_3 are the components of the unit vector $\boldsymbol{\gamma}$ fixed in space in the direction of the axis of symmetry of the force fields applied to the body, all being referred to the principal axes of inertia at the fixed point. The potential V and the vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ depend only on the Poisson variables γ_1, γ_2, γ_3.

As was shown in [1], in their general form different terms of equations (1) may be interpreted in one or more of the following ways.

The potential V can be understood as the result of the scalar interactions of a gravitational field with the mass distribution in the body, an electric field with a permanent

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distribution of electric charges and a magnetic field with some magnetized parts or steady currents in electric circuits on the body. A constant term of the vector $\boldsymbol{\mu}$ is the so-called gyrostatic moment that appears when the body carries a symmetric rotor while the variable terms appear as a result of the Lorentz effect of the magnetic field on the electric charges. The gyrostatic moment can also be due to internal cyclic degrees of freedom such as the circulation of fluid in holes inside the body or forced stationary motions such as motors and the flow of fluids in circuits in the body.

An interesting problem when the body has no point of it fixed at all times but nevertheless has the same form of equations of motion is the problem of motion by inertia of a body, whose surface may be simply or multi-connected, in an infinitely extending ideal incompressible fluid (for a derivation of the equations of motion see [2]). This problem has been intensively studied, but mostly in complete isolation from other problems of rigid body dynamics. In particular, except for some limited analogies noted by Steklov [3] and Kharlamov [4](see also [5]) it was not known that the problem in its general form can be formulated as a Lagrangian or Hamiltonian system. In our work [6], the general problem was brought to a special version of the form (1). This means that the forces exerted by the fluid on the body can be replaced, with respect to their effect on the rotational motion of the body, by a set of gravitational and electromagnetic interactions. In particular, this analogy has allowed the construction of solutions for the problem of motion of an electrified and magnetized body from the known solutions of a classical problem [6] and, on the other hand, allowed the application of all the principles of analytical mechanics to that classical problem [7].

Equations (1) admit three general first integrals:

(a) Jacobi's integral $I_1 = \frac{1}{2}Ap^2 + \frac{1}{2}Bq^2 + \frac{1}{2}Cr^2 + V$.

(b) The geometric integral $I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$.

(c) An integral linear in the components of angular velocity I_3 (say), corresponding to the vanishing moment of forces around the axis of symmetry of the fields. For simplicity this integral will not be written in the general case but separately for each of the cases considered below.

Thus, according to Jacobi's theorem of the last integrating multiplier, the system (1) will be integrable and its solution can be reduced to quadratures whenever one more integral I_4 is found [8].

On the other hand, the system (1) can be reduced on any arbitrary level of the cyclic integral I_3 to equivalent Lagrangian and Hamiltonian forms (see, for example, [1] about the Lagrangian corresponding to (1)) as a system with two degrees of freedom which admits one integral I_1 ($I_2 = 1$ reduces to an identity in any set of generalized coordinates). The system will be completely integrable in the sense of Liouville (see for example [8, 9]) whenever the integral I_4 exists and is functionally independent of I_1, I_3.

In the rest of the present paper we present six new general integrable cases of the system (1) that generalize most of the known results on the subject by the introduction of several parameters and thus reveal the striking richness of this system. For each case we briefly give the necessary forms of the potential V, the vector $\boldsymbol{\mu}$ and the integrals I_3 and I_4. In every case the constancy of those integrals can be verified by the reader in a straightforward way. We also give a summary of previous results which can be obtained from each one of our cases as special versions. Whenever met, unless otherwise stated, the symbols n_1, n_2, n_3, n, N, s_1, s_2, s_3, a, b, c, c_1, c_2, c_3, a_1, a_2 and ε signify arbitrary real constants.

The results of the present paper were obtained by the method of invariance of the equations of motion under certain transformations. A simple version of this method, corresponding to a uniform rotation transformation was introduced in [14] and [1] and has proved useful in constructing new integrable cases of a charged body and of a body in a liquid [6, 14]. The possibility of generalizing this method to include position-dependent rotation transformation was pointed out in [7]. The method we used to obtain the new cases has also led to the construction of fifteen conditional integrable cases, valid only on a single level of I_3, and a larger number of particular solutions generalizing known ones. It also enables the construction of the explicit solution of the equations of motion in many cases. Those results will be presented in a forthcoming paper.

2. The first case

This case is valid for a body with arbitrary moments of inertia. If in (1) we choose

$$V = (A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)\{b - \tfrac{1}{2}[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]^2\} \tag{2}$$

$$\begin{aligned}
\mu_1 &= \gamma_1\{(A - B - C)[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)] \\
&\quad + 2n_1[A(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\} \\
\mu_2 &= \gamma_2\{(B - C - A)[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)] \\
&\quad + 2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\} \\
\mu_3 &= \gamma_3\{(C - A - B)[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)] \\
&\quad + 2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}
\end{aligned} \tag{3}$$

the system (1) will be completely integrable with

$$I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)](A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \tag{4}$$

$$\begin{aligned}
I_4 &= \tfrac{1}{2}A^2\{p + \gamma_1[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2 \\
&\quad + \tfrac{1}{2}B^2\{q + \gamma_2[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2 \\
&\quad + \tfrac{1}{2}C^2\{r + \gamma_3[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2 \\
&\quad - \{b - n_1[Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 \\
&\quad + [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)](A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\} \\
&\quad \times (BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2).
\end{aligned} \tag{5}$$

We first note that the fourth integral is quadratic in velocities and contains linear terms. If here we set $n_1 = 0$, this result becomes equivalent to the case due to Clebsch [11] in the problem of a body in a liquid. If, further, we put $n = 0$, we obtain a case which has, in addition to the previous interpretation, two physically different interpretations.

(a) The case of motion of a rigid body about a fixed point under the action of the approximate field of a distant Newtonian centre of attraction [12] (see also [10]).

(b) Brun's problem of the motion of a body about a fixed point under the assumption that each element of the body is attracted to a fixed plane passing through the fixed point by a force that is proportional to the distance to that plane [13].

If in (2)–(5) we put $n_1 = b = 0$, we get a case that was found in [14] which generalizes an earlier case of motion of a charged body found by Grioli [15]. If further we put $n = 0$, we get the basic case due to Euler.

3. The second case

This is a case of complete dynamical symmetry $B = C = A$. Let

$$V = \tfrac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2) - \tfrac{1}{2}A(n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)^2 \tag{6}$$

$$\mu_1 = -A\gamma_1[n + n_1\gamma_1^2 + \gamma_2^2(3n_2 - 2n_1) + \gamma_3^2(3n_3 - 2n_1)]$$

$$\mu_2 = -A\gamma_2[n + \gamma_1^2(3n_1 - 2n_2) + n_2\gamma_2^2 + (3n_3 - 2n_2)\gamma_3^2]$$
$$\mu_3 = -A\gamma_3[n + \gamma_1^2(3n_1 - 2n_3) + \gamma_2^2(3n_2 - 2n_3) + n_3\gamma_3^2] \tag{7}$$

then

$$I_3 = A(p\gamma_1 + q\gamma_2 + r\gamma_3 + n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2) \tag{8}$$

$$\begin{aligned} I_4 = A((c_1 - 2n_1 I_3)&[p + (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\gamma_1]^2 \\ &+(c_2 - 2n_2 I_3)[q + (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\gamma_2]^2 \\ &+(c_3 - 2n_3 I_3)[r + (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\gamma_3]^2) \\ &-(2n_2 I_3 - c_2)(2n_3 I_3 - c_3)\gamma_1^2 - (2n_3 I_3 - c_3)(2n_1 I_3 - c_1)\gamma_2^2 \\ &-(2n_1 I_3 - c_1)(2n_2 I_3 - c_2)\gamma_3^2. \end{aligned} \tag{9}$$

Note that in (9) I_3 stands for its expression (8). In the general case, the fourth integral is a polynomial of the third degree in the velocities with leading coefficients depending on γ. In the special case when $n_1 : n_2 : n_3 :: c_1 : c_2 : c_3$, a constant factor can be dropped out and the integral becomes of the second degree.

The present case generalizes by the introduction of the three parameters n_1, n_2, n_3 the case of motion of a body in liquid known as Clebsch's case of complete dynamical symmetry.

4. The third case

It is also a case of complete dynamical symmetry $B = C = A$. For it we have

$$\begin{aligned} V = s_1\gamma_1 + s_2\gamma_2 + s_3\gamma_3 - \frac{1}{2}\frac{abc}{A}\left(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}\right) - \frac{1}{2}A(n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3)^2 \\ +\frac{1}{2}(n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3)[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2] \end{aligned} \tag{10}$$

$$\mu_1 = An_1 + \gamma_1[a - An + 2A(n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3)]$$
$$\mu_2 = An_2 + \gamma_2[b - An + 2A(n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3)]$$
$$\mu_3 = An_3 + \gamma_3[c - An + 2A(n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3)] \tag{11}$$

and

$$\begin{aligned} I_3 = A(p\gamma_1 + q\gamma_2 + \gamma_3 r) &- \tfrac{1}{2}[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2] \\ &+A(n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3) \end{aligned} \tag{12}$$

$$\begin{aligned} I_4 = \tfrac{1}{2}A\{(b+c)&[p + (n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n)\gamma_1]^2 \\ &+(c+a)[q + (n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n)\gamma_2]^2 \\ &+(a+b)[r + (n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 + n)\gamma_3]^2\} \\ &+(-n_1 I_3 + s_1)[A(p + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_1) + a\gamma_1] \\ &+(-n_2 I_3 + s_2)[A(q + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_2) + b\gamma_2] \\ &+(-n_3 I_3 + s_3)[A(r + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_3) + c\gamma_3] \\ &-abc[(p + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_1)\gamma_1/a \\ &+(q + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_2)\gamma_2/b \\ &+(r + (n_1\gamma_1 + n_2\gamma_2 + \gamma_3 n_3 + n)\gamma_3)\gamma_3/c]. \end{aligned} \tag{13}$$

The fourth integral is a polynomial of the second degree in velocities with leading coefficients depending on γ. For $n_1 = n_2 = n_3 = 0$, our case becomes equivalent to a case found by Rubanovsky [16] in the dynamics of a body in liquid, which, in turn, generalizes an earlier result in the same subject due to Lyapunov [17].

5. The fourth case

In this case the body has the famous Kovalevskaya configuration $A = B = 2C$ and

$$V = C(a_1\gamma_1 + a_2\gamma_2) - Ck\gamma_3(n + n_1\gamma_1 + n_2\gamma_2) - \tfrac{1}{2}C(n + n_1\gamma_1 + n_2\gamma_2)^2(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2) \tag{14}$$

$$\begin{aligned}
\mu_1 &= C(-n\gamma_1 - n_1\gamma_1^2 + 2n_1\gamma_2^2 + n_1\gamma_3^2 - 3n_2\gamma_1\gamma_2)\\
\mu_2 &= C(-\gamma_2 n + 2n_2\gamma_1^2 - n_2\gamma_2^2 + n_2\gamma_3^2 - 3n_1\gamma_1\gamma_2)\\
\mu_3 &= C(k - 3n\gamma_3 - 5n_1\gamma_1\gamma_3 - 5n_2\gamma_2\gamma_3)
\end{aligned} \tag{15}$$

$$I_3 = C(2\gamma_1 p + 2q\gamma_2 + r\gamma_3) + Ck\gamma_3 + C(n + n_1\gamma_1 + n_2\gamma_2)(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2) \tag{16}$$

$$\begin{aligned}
I_4 = \Bigg\{ & p + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_1|^2 - q + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_2|^2\\
& - \left(a_1 - n_1\frac{I_3}{C}\right)\gamma_1 + \left(a_2 - n_2\frac{I_3}{C}\right)\gamma_2\Bigg\}^2\\
& + \Bigg\{2(p + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_1)(q + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_2)\\
& - \left(a_1 - n_1\frac{I_3}{C}\right)\gamma_2 - \left(a_2 - n_2\frac{I_3}{C}\right)\gamma_1\Bigg\}^2 + 2k(r + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_3 - k)\\
& \times[(p + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_1)^2 + (q + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_2)^2]\\
& - 4k\gamma_3\Bigg\{\left(a_1 - n_1\frac{I_3}{C}\right)[p + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_1]\\
& + \left(a_2 - n_2\frac{I_3}{C}\right)[q + (n + n_1\gamma_1 + n_2\gamma_2)\gamma_2]\Bigg\}.
\end{aligned} \tag{17}$$

As in the classical Kovalevskaya's case, the fourth integral has the fourth degree in velocities. The present result generalizes Kovalevskaya's by including four physically significant parameters k, n, n_1, n_2. It also generalizes some earlier results of the present author. For $n_1 = n_2 = 0$, we get the case of a body in a liquid [6] and if, moreover, $n = 0$, we get the case of a heavy gyrostat found in [18] (see also [19])

6. The fifth case. A case of singular potential

The present case is, like the preceding one, valid for $A = B = 2C$. For it we have

$$V = C(a_1\gamma_1 + a_2\gamma_2) + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} - \frac{1}{2}C\left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)^2(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2) \tag{18}$$

$$\begin{aligned}
\mu_1 &= C\left(-n\gamma_1 - n_1\gamma_1^2 + 2n_1\gamma_2^2 + n_1\gamma_3^2 - 3n_2\gamma_1\gamma_2 + \frac{N\gamma_1}{(1-\gamma_3^2)^{\frac{3}{2}}}\right)\\
\mu_2 &= C\left(-n\gamma_2 + 2n_2\gamma_1^2 - n_2\gamma_2^2 + n_2\gamma_3^2 - 3n_1\gamma_1\gamma_2 + \frac{N\gamma_2}{(1-\gamma_3^2)^{\frac{3}{2}}}\right)\\
\mu_3 &= -C\gamma_3\left(3n + 5n_1\gamma_1 + 5n_2\gamma_2 + \frac{N\gamma_3}{\sqrt{1-\gamma_3^2}}\right)
\end{aligned} \tag{19}$$

$$I_3 = C(2p\gamma_1 + 2q\gamma_2 + r\gamma_3) + C\left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2) \quad (20)$$

$$\begin{aligned} I_4 = \Bigg\{ & \left[p + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_1\right]^2 \\ & - \left[q + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_2\right]^2 \\ & - \left(a_1 - n_1\frac{I_3}{C}\right)\gamma_1 + \left(a_2 - n_2\frac{I_3}{C}\right)\gamma_2 \Bigg\}^2 \\ & + \Bigg\{\left[p + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_1\right] \\ & \times \left[q + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_2\right] \\ & - \left(a_1 - n_1\frac{I_3}{C}\right)\gamma_2 - \left(a_2 - n_2\frac{I_3}{C}\right)\gamma_1 \Bigg\}^2 + 2\left(\varepsilon - \frac{NI_3}{C}\right) \\ & \times \Bigg\{\left[p + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_1\right]^2 \\ & + \left[q + \left(n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}\right)\gamma_2\right]^2 \Bigg\} \left\{\sqrt{1-\gamma_3^2}\right\}^{-1} \\ & + \frac{(\varepsilon - (NI_3/C))^2}{1-\gamma_3^2}. \end{aligned} \quad (21)$$

The case $n_1 = n_2 = N = 0$ was found in [20]. The case $n = n_1 = n_2 = N = 0$ was found in [21]. If, moreover, $\varepsilon = 0$, we get the original case of Kovalevskaya.

7. Generalization of Lagrange's case

In the problems considered above in the dynamics of rigid bodies there are only two cases that were not generalized by the above results. Namely, Lagrange's case of a heavy top and Steklov's case of a body in a liquid.

Lagrange's case admits a simple generalization that contains three arbitrary functions. In fact, if we choose

$$B = A,\ V = V(\gamma_3) \qquad \boldsymbol{\mu} = (\gamma_1 F'(\gamma_3), \gamma_2 F'(\gamma_3), -F(\gamma_3) + (\gamma_1^2 + \gamma_2^2)G'(\gamma_3) - 2\gamma_3 G(\gamma_3))$$

where F and G are arbitrary functions of γ_3 the system (1) admits the integrals:

$$I_3 = A(p\gamma_1 + q\gamma_2) + \gamma_3(Cr - F(\gamma_3)) + (\gamma_1^2 + \gamma_2^2)G(\gamma_3) \qquad I_4 = Cr - F(\gamma_3).$$

Thus, of all the known results only the case of a body in liquid found by Rubanovsky [16] remains without generalization. This case includes as special versions an earlier case due to Steklov [3] and the Jonkovsky–Volterra case of a gyrostat moving by inertia [22, 23].

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